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## Incomplete Integrals of Cylindrical Functions

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# INCOMPLETE INTEGRALS OF CYLINDRICAL FUNCTIONS

## INTRODUCTION

The class of cylindrical functions  $C$  includes Bessel functions of the first kind  $J$ , modified Bessel functions  $I$ , Bessel functions of the second kind or Neumann functions  $Y$  (or  $N$ ), Bessel functions of imaginary argument or MacDonald functions  $K$ , and Bessel functions of the third kind that include Hankel functions of the first and second kind,  $H^{(1)}$  and  $H^{(2)}$ .

The general incomplete Lipschitz-Hankel integral of cylindrical functions  $C_\nu(z)$  is defined as the function of two complex variables:

$$C_{e_\nu}(a, z) := \int_0^z e^{at} t^\mu C_\nu(t) dt. \quad (1)$$

Here the symbol  $e$  denotes the presence of the exponential function and  $\mu, \nu$  may be complex. Analogously, we define integrals that contain the functions  $\sin(at)$  and  $\cos(at)$  in place of  $\exp(at)$ :

$$C_{s_\nu}(a, z) := \int_0^z \sin(at) t^\mu C_\nu(t) dt \quad (2)$$

$$C_{c_\nu}(a, z) := \int_0^z \cos(at) t^\mu C_\nu(t) dt. \quad (3)$$

To assure convergence of  $C_{e_\nu}(a, z)$  and  $C_{c_\nu}(a, z)$ , it is necessary that  $\operatorname{Re}(\mu + 1) > |\operatorname{Re} \nu|$  when  $C = K, Y, H^{(1)}, H^{(2)}$ ;  $\operatorname{Re}(1 + \mu + \nu) > 0$  when  $C = I, J$ . For convergence of  $C_{s_\nu}(a, z)$  replace  $\mu$  by  $\mu + 1$  in the latter two inequalities. When  $\mu = \nu$ , we define, for example,  $C_{e_\nu} := C_\nu$ , where for convergence  $\operatorname{Re} \mu > -1/2$  for all  $C$ .

Integrals of the type given by Eqs. (1) to (3) occur very often in applied mathematics. Agrest and Maksimov [1] have found representations for  $C_{e_\nu}(a, z)$ ,  $C_{s_\nu}(a, z)$ , and  $C_{c_\nu}(a, z)$  using incomplete cylindrical functions which are themselves expressed as integrals. In this report which is an extension and revision of previous work [2] we give representations for  $C_{e_\nu}(a, z)$ ,  $C_{s_\nu}(a, z)$ , and  $C_{c_\nu}(a, z)$  using only the Kampé de Fériet double hypergeometric functions  $F_{2:1;0}^{0:2;1}[x, y]$  and  $F_{1:1;0}^{1:0;1}[x, y]$ .

## PRELIMINARY RESULTS AND DEFINITIONS

To begin, we summarize some results that are found in Ref. 3, p. 85: Let  $a$  and  $b$  be arbitrary constants,

$$\mathbf{F}_\nu(z) := aI_\nu(z) + b e^{i\nu\pi} K_\nu(z)$$

$$\mathbf{G}_\nu(z) := aJ_\nu(z) + bY_\nu(z)$$

$$\alpha := \begin{cases} i: \mathbf{H} = \mathbf{F} \\ 1: \mathbf{H} = \mathbf{G} \end{cases} \quad \beta := \begin{cases} 1: \mathbf{H} = \mathbf{F} \\ 0: \mathbf{H} = \mathbf{G} \end{cases}$$

Then

$$\int_0^z t^\mu H_\nu(t) dt = e^{-\frac{\pi i \beta \mu}{2}} [(\mu + \nu - 1)z H_\nu(z) s_{\mu-1, \nu-1}(\alpha z) + (2\beta - 1)\alpha z H_{\nu-1}(z) s_{\mu, \nu}(\alpha z)], \quad (4)$$

where the Lommel functions  $s_{\mu, \nu}$  ( $\mu \pm \nu \neq -1, -3, -5, \dots$ ) are given by

$$s_{\mu, \nu}(z) = \frac{z^{\mu+1}}{(\mu - \nu + 1)(\mu + \nu + 1)} {}_1F_2 \left[ 1; \frac{\mu - \nu + 3}{2}, \frac{\mu + \nu + 3}{2}; \frac{-z^2}{4} \right]. \quad (5)$$

Now defining

$$\xi: = \begin{cases} 1: C = I, K \\ -1: C = H, J, Y \end{cases} \quad \eta: = \begin{cases} 1: C = K \\ -1: C = H, I, J, Y \end{cases} \quad \omega: = \begin{cases} 1: C = K \\ 2/\pi: C = H, Y \\ 0: C = I, J \end{cases}$$

we may deduce from Eqs. (4) and (5) the result

$$\begin{aligned} \int_0^z t^\mu C_\nu(t) dt &= \frac{z^{\mu+1}}{\mu - \nu + 1} \left\{ C_\nu(z) {}_1F_2 \left[ 1; \frac{\mu - \nu + 3}{2}, \frac{\mu + \nu + 1}{2}; \frac{\xi z^2}{4} \right] \right. \\ &\quad \left. + \frac{\eta z C_{\nu-1}(z)}{\mu + \nu + 1} {}_1F_2 \left[ 1; \frac{\mu - \nu + 3}{2}, \frac{\mu + \nu + 3}{2}; \frac{\xi z^2}{4} \right] \right\}. \end{aligned} \quad (6)$$

We define the Kampé de Fériet double hypergeometric functions  $L$ ,  $M$  and  $Q$  and give associated generating relations [4, 5]:

$$\begin{aligned} L[\alpha, \beta; \gamma, \delta; x, y] &= F_{2:0;0}^{0:1;1} \left[ \begin{matrix} \alpha, \beta; & x, y \\ \gamma, \delta; & \end{matrix} \right] \quad |x| < \infty, \quad |y| < \infty \\ M[\alpha, \beta; \gamma, \delta; x, y] &= F_{1:1;0}^{1:0;1} \left[ \begin{matrix} \alpha; \quad \beta; & x, y \\ \gamma; \quad \delta; & \end{matrix} \right] \quad |x| < \infty, \quad |y| < 1 \\ Q[\alpha, \beta, \gamma; \mu, \nu, \lambda; x, y] &= F_{2:1;0}^{0:2;1} \left[ \begin{matrix} \alpha, \beta; \quad \gamma; & x, y \\ \mu, \nu; \quad \lambda; & \end{matrix} \right] \quad |x| < \infty, \quad |y| < \infty \\ L[\alpha, \beta; \gamma, \delta; x, y] &= \sum_{m=0}^{\infty} \frac{(\alpha)_m}{(\gamma)_m (\delta)_m} \frac{x^m}{m!} {}_1F_2 [\beta; m + \gamma, m + \delta; y] \end{aligned} \quad (7)$$

$$M[\alpha, \beta; \gamma, \delta; x, t] = \sum_{m=0}^{\infty} \frac{(\alpha)_m}{(\gamma)_m (\delta)_m} \frac{x^m}{m!} {}_3F_0[\beta, -m, 1-\delta-m; -; t] \quad (8)$$

$$Q[\alpha, \beta, \gamma; \mu, \nu, \lambda; x, y] = \sum_{m=0}^{\infty} \frac{(\alpha)_m (\beta)_m}{(\mu)_m (\nu)_m (\lambda)_m} \frac{x^m}{m!} {}_1F_2[\gamma; m+\mu, m+\nu; y]. \quad (9)$$

It is easy to see that the function  $L$  is a special case of  $Q$ :

$$Q[\alpha, \lambda, \beta; \gamma, \delta, \lambda; x, y] = L[\alpha, \beta; \gamma, \delta; x, y].$$

For brevity we define the parameter lists

$$A_1(\mu, \nu) := \frac{\mu + \nu + 1}{2}, \frac{\mu - \nu + 1}{2}, 1; \frac{\mu + \nu + 1}{2}, \frac{\mu - \nu + 3}{2}, \frac{1}{2}$$

$$A_2(\mu, \nu) := \frac{\mu + \nu + 1}{2}, \frac{\mu - \nu + 1}{2}, 1; \frac{\mu + \nu + 3}{2}, \frac{\mu - \nu + 3}{2}, \frac{1}{2}$$

$$B_1(\mu, \nu) := \frac{\mu + \nu + 2}{2}, \frac{\mu - \nu + 2}{2}, 1; \frac{\mu + \nu + 2}{2}, \frac{\mu - \nu + 4}{2}, \frac{3}{2}$$

$$B_2(\mu, \nu) := \frac{\mu + \nu + 2}{2}, \frac{\mu - \nu + 2}{2}, 1; \frac{\mu + \nu + 4}{2}, \frac{\mu - \nu + 4}{2}, \frac{3}{2}$$

$$D_1(\mu) := \frac{1}{2} + \mu, 1; \frac{1}{2} + \mu, \frac{3}{2} \quad G_1(\mu) := 1 + \mu, 1; \frac{3}{2}, \mu$$

$$D_2(\mu) := \frac{1}{2} + \mu, 1; \frac{3}{2} + \mu, \frac{3}{2} \quad G_2(\mu) := 1 + \mu, 1; \frac{3}{2}, 1 + \mu$$

$$E_1(\mu, \nu) := \frac{1 + \mu + \nu}{2}, \frac{2 + \mu + \nu}{2}, 1; \frac{2 + \mu + \nu}{2}, \frac{3 + \mu + \nu}{2}, 1 + \nu$$

$$E_2(\mu, \nu) := \frac{1 + \mu + \nu}{2}, \frac{2 + \mu + \nu}{2}, 1; \frac{3 + \mu + \nu}{2}, \frac{4 + \mu + \nu}{2}, 1 + \nu$$

$$F_1(\mu) := \frac{1}{2} + \mu, 1; 1 + \mu, \frac{3}{2} + \mu \quad H_1(\mu) := 2 + \mu, 1, 1; 2 + \mu, 3, \frac{5}{2}$$

$$F_2(\mu) := \frac{1}{2} + \mu, 1; 2 + \mu, \frac{3}{2} + \mu \quad H_2(\mu) := 2 + \mu, 1, 1; 3 + \mu, 3, \frac{5}{2}$$

## REPRESENTATIONS FOR $C_{e_n}(a, z)$ , $C_{s_n}(a, z)$ , $C_{c_n}(a, z)$

Substituting the Maclaurin series for  $\exp(at)$  in Eq. (1) and splitting into even and odd terms we obtain on integrating term by term

$$C_{e_n}(a, z) = \sum_{n=0}^{\infty} \frac{a^{2n}}{(2n)!} \int_0^z t^{\mu+2n} C_\nu(t) dt + \sum_{n=0}^{\infty} \frac{a^{1+2n}}{(1+2n)!} \int_0^z t^{1+\mu+2n} C_\nu(t) dt. \quad (10)$$

Then using Eq. (6) and the generating relation Eq. (9) we obtain after a tedious but straightforward computation the result

$$\begin{aligned} C_{e_n}(a, z) = & z^{1+\mu} C_\nu(z) \left\{ \frac{1}{\mu-\nu+1} Q \left[ A_1; \frac{a^2 z^2}{4}, \frac{\xi z^2}{4} \right] + \frac{az}{\mu-\nu+2} Q \left[ B_1; \frac{a^2 z^2}{4}, \frac{\xi z^2}{4} \right] \right\} \\ & + \eta z^{2+\mu} C_{\nu-1}(z) \left\{ \frac{1}{(\mu+\nu+1)(\mu-\nu+1)} Q \left[ A_2; \frac{a^2 z^2}{4}, \frac{\xi z^2}{4} \right] \right. \\ & \left. + \frac{az}{(\mu+\nu+2)(\mu-\nu+2)} Q \left[ B_2; \frac{a^2 z^2}{4}, \frac{\xi z^2}{4} \right] \right\}. \end{aligned} \quad (11)$$

Since

$$C_{s_n}(a, z) = \frac{1}{2i} \left\{ C_{e_n}(ia, z) - C_{e_n}(-ia, z) \right\}$$

$$C_{c_n}(a, z) = \frac{1}{2} \left\{ C_{e_n}(ia, z) + C_{e_n}(-ia, z) \right\}$$

we may write

$$\begin{aligned} C_{s_n}(a, z) = & \frac{az^{2+\mu}}{\mu-\nu+2} \left\{ C_\nu(z) Q \left[ B_1; \frac{-a^2 z^2}{4}, \frac{\xi z^2}{4} \right] \right. \\ & \left. + \frac{\eta z}{\mu+\nu+2} C_{\nu-1}(z) Q \left[ B_2; \frac{-a^2 z^2}{4}, \frac{\xi z^2}{4} \right] \right\} \end{aligned} \quad (12)$$

$$\begin{aligned} C_{c_n}(a, z) = & \frac{z^{1+\mu}}{\mu-\nu+1} \left\{ C_\nu(z) Q \left[ A_1; \frac{-a^2 z^2}{4}, \frac{\xi z^2}{4} \right] \right. \\ & \left. + \frac{\eta z}{\mu+\nu+1} C_{\nu-1}(z) Q \left[ A_2; \frac{-a^2 z^2}{4}, \frac{\xi z^2}{4} \right] \right\}. \end{aligned} \quad (13)$$

For  $\mu = \nu$ , Eqs. (11) to (13) reduce to

$$C_{e_\mu}(a, z) = z^{1+\mu} C_\mu(z) \left\{ L \left[ D_1; \frac{a^2 z^2}{4}, \frac{\xi z^2}{4} \right] + \frac{az}{2} Q \left[ B_1(\mu, \mu); \frac{a^2 z^2}{4}, \frac{\xi z^2}{4} \right] \right\}$$

$$+ \eta z^{2+\mu} C_{\mu-1}(z) \left\{ \frac{1}{1+2\mu} L \left[ D_2; \frac{a^2 z^2}{4}, \frac{\xi z^2}{4} \right] + \frac{az}{4(1+\mu)} Q \left[ B_2(\mu, \mu); \frac{a^2 z^2}{4}, \frac{\xi z^2}{4} \right] \right\} \quad (14)$$

$$C_{s_\mu}(a, z) = \frac{1}{2} az^{2+\mu} \left\{ C_\mu(z) Q \left[ B_1(\mu, \mu); \frac{-a^2 z^2}{4}, \frac{\xi z^2}{4} \right] \right.$$

$$\left. + \frac{\eta z}{2(1+\mu)} C_{\mu-1}(z) Q \left[ B_2(\mu, \mu); \frac{-a^2 z^2}{4}, \frac{\xi z^2}{4} \right] \right\} \quad (15)$$

$$C_c(a, z) = z^{1+\mu} \left\{ C_\mu(z) L \left[ D_1; \frac{-a^2 z^2}{4}, \frac{\xi z^2}{4} \right] + \frac{\eta z}{1+2\mu} C_{\mu-1}(z) L \left[ D_2; \frac{-a^2 z^2}{4}, \frac{\xi z^2}{4} \right] \right\}. \quad (16)$$

Defining  $J^+ := J$ ,  $J^- := I$ , it is interesting to note that we may also write [7]

$$J_{e_\mu}^\pm(a, z) = \frac{z^{1+\mu+\nu} e^{\pm az}}{2^\nu (1+\mu+\nu) \Gamma(1+\nu)} \left\{ Q \left[ E_1; \frac{\mp z^2}{4}, \frac{a^2 z^2}{4} \right] - \frac{az}{2+\mu+\nu} Q \left[ E_2; \frac{\mp z^2}{4}, \frac{a^2 z^2}{4} \right] \right\}$$

$$(17)$$

$$J_{e_\mu}^\pm(a, z) = \frac{z(z^2/2)^\mu e^{\pm az}}{(1+2\mu) \Gamma(1+\mu)} \left\{ L \left[ F_1; \frac{\mp z^2}{4}, \frac{a^2 z^2}{4} \right] - \frac{az}{2(1+\mu)} L \left[ F_2; \frac{\mp z^2}{4}, \frac{a^2 z^2}{4} \right] \right\}. \quad (18)$$

Here the Bessel functions  $J_\nu^\pm$  do not appear.

We remark that Eqs. (11) to (18) are valid for  $|a| < \infty$ ,  $|z| < \infty$ .

### OTHER REPRESENTATIONS FOR $C_{e_\mu}(a, z)$ , $C_{s_\mu}(a, z)$ , $C_c(a, z)$

In Eq. (4)  $s_{\mu, \nu}$  may be replaced by the Lommel function  $S_{\mu, \nu}$  where, when either of the numbers  $\mu \pm \nu$  is an odd positive integer,

$$S_{\mu, \nu}(z) = z^{\mu-1} {}_3F_0 \left[ 1, \frac{\nu-\mu+1}{2}, -\frac{\nu+\mu-1}{2}; \frac{1}{z}; \frac{-4}{z^2} \right]. \quad (19)$$

Defining

$$\omega_n := \begin{cases} 1 & C = K \\ (-1)^n 2/\pi & C = H, Y \\ 0 & C = I, J \end{cases}$$

and replacing  $s_{\mu, \nu}$  in Eq. (4) by  $S_{\mu, \nu}$  given by Eq. (19) we obtain for nonnegative integers  $n$  the result

$$\begin{aligned} \int_0^z t^{\nu + 2n + 1} C_\nu(t) dt &= \omega_n 2^{\nu + 2n} \Gamma(n + 1) \Gamma(\nu + n + 1) \\ &- \xi z^{\nu + 2n} \{ 2(\nu + n) C_\nu(z) {}_3F_0[1, -n, 1 - \nu - n; -; 4\xi/z^2] \\ &+ \eta z C_{\nu-1}(z) {}_3F_0[1, -n, -\nu - n; -; 4\xi/z^2] \}. \end{aligned} \quad (20)$$

Using Eqs. (6) and (20) together with Eq. (10) and noting the generating relations Eqs. (7) and (8) we arrive at

$$\begin{aligned} C_{\epsilon_0}(a, z) &= \omega 2^\mu \Gamma(1 + \mu) a {}_2F_1 \left[ 1, 1 + \mu; \frac{3}{2}; \xi a^2 \right] \\ &+ z^\mu C_\mu(z) \left\{ z L \left[ D_1(\mu); \frac{a^2 z^2}{4}, \frac{\xi z^2}{4} \right] - 2\xi a \mu M \left[ G_1(\mu); \frac{a^2 z^2}{4}, \xi a^2 \right] \right\} \\ &+ \eta z^{1+\mu} C_{\mu-1}(z) \left\{ \frac{z}{1+2\mu} L \left[ D_2(\mu); \frac{a^2 z^2}{4}, \frac{\xi z^2}{4} \right] - \xi a M \left[ G_2(\mu); \frac{a^2 z^2}{4}, \xi a^2 \right] \right\}, \\ |a| < 1, \quad \operatorname{Re} \mu > -1/2. \end{aligned} \quad (21)$$

Since this computation is straightforward, it is omitted.

In order to obtain formulas for the  $C_{\epsilon_0}(a, z)$ , i.e. for  $\mu = 0$ , from Eq. (21) we may readily show that

$$\lim_{\delta \rightarrow 0} \delta M[\alpha, \beta; \gamma, \delta; x, y] = \frac{\alpha x}{\gamma} M[\alpha + 1, \beta; \gamma + 1, 2; x, y]$$

from which it follows that

$$\lim_{\mu \rightarrow 0} 2\mu M \left[ G_1(\mu); \frac{a^2 z^2}{4}, \xi a^2 \right] = \frac{1}{3} a^2 z^2 M \left[ 2, 1; \frac{5}{2}, 2; \frac{a^2 z^2}{4}, \xi a^2 \right].$$

We also note that when  $\mu = 0$  in Eq. (21),

$${}_2F_1 \left[ 1, 1; \frac{3}{2}; a^2 \right] = \frac{\sin^{-1} a}{a \sqrt{1 - a^2}}$$

so that also

$${}_2F_1 \left[ 1, 1; \frac{3}{2}; -a^2 \right] = \frac{\sinh^{-1} a}{a \sqrt{1 + a^2}}.$$

The result obtained from Eq. (21) for  $K_{\epsilon_0}(a, z)$  is given in [4].

On setting  $\mu = \nu$  in the equations for  $C_{s_\mu}$  and  $C_{c_\mu}$  given in the previous section in terms of  $C_{\epsilon_\mu}$  we obtain on using Eq. (21)

$$\begin{aligned} C_{s_\mu}(a, z) &= \omega a^{2\mu} \Gamma(1 + \mu) {}_2F_1 \left[ 1, 1 + \mu; \frac{3}{2}; -\xi a^2 \right] \\ &\quad - \xi a z^\mu \left\{ 2\mu C_\mu(z) M \left[ G_1; \frac{-a^2 z^2}{4}, -\xi a^2 \right] + \eta z C_{\mu-1}(z) M \left[ G_2; \frac{-a^2 z^2}{4}, -\xi a^2 \right] \right\}, \end{aligned} \quad (22)$$

$$|a| < 1, \quad \operatorname{Re} \mu > -1.$$

For  $C_{c_\mu}(a, z)$  we obtain again the result given by Eq. (16).

It is shown in Ref. 5 that

$$\begin{aligned} M[\alpha, 1; \gamma, \delta; tx, t] &= 1 + {}_0F_1[-; \delta; x] \{{}_2F_1[\alpha, 1; \gamma; t] - 1\} \\ &\quad - \frac{\alpha t x^2}{2\gamma\delta(\delta+1)} Q[1 + \alpha, 1, 1; 2 + \delta, 3, 1 + \gamma; tx, x]. \end{aligned} \quad (23)$$

This equation provides the corollary that  $M[\alpha, 1; \gamma, \delta; tx, t]$  converges on the unit circle  $|t| = 1$  if and only if  ${}_2F_1[\alpha, 1; \gamma; t]$  does. We then deduce that Eqs. (21) and (22) are conditionally convergent on  $|a| = 1, \pm \xi a^2 \neq 1$  provided that  $|\operatorname{Re} \mu| < 1/2$ .

We may, however, use Eq. (23) to better advantage. We find

$$\begin{aligned} M \left[ G_1(\nu); \frac{a^2 z^2}{4}, \xi a^2 \right] &= 1 + \left( \frac{2}{z} \right)^{\nu-1} \Gamma(\nu) J_{\nu-\frac{\xi}{2}-1}(z) \left\{ {}_2F_1 \left[ 1 + \nu, 1; \frac{3}{2}; \xi a^2 \right] - 1 \right\} \\ &\quad - \frac{\xi a^2 (z^2/4)^2}{3\nu} Q \left[ H_1(\nu); \frac{a^2 z^2}{4}, \frac{\xi z^2}{4} \right] \end{aligned}$$

$$M \left[ G_2(\nu); \frac{a^2 z^2}{4}, \xi a^2 \right] = 1 + \left( \frac{2}{z} \right)^\nu \Gamma(1 + \nu) J_\nu^{-\xi}(z) \left\{ {}_2F_1 \left[ 1 + \nu, 1; \frac{3}{2}; \xi a^2 \right] - 1 \right\} \\ - \frac{\xi a^2 (z^2/4)^2}{3(2 + \nu)} Q \left[ H_2(\nu); \frac{a^2 z^2}{4}, \frac{\xi z^2}{4} \right]$$

where we agree that  $J^{\pm 1} = J^\pm$ .

Using these equations together with Eq. (21) and noting the Wronskian relations

$$K_{\nu+1}(z)I_\nu(z) + K_\nu(z)I_{\nu+1}(z) = 1/z$$

$$J_{\nu+1}(z)Y_\nu(z) - J_\nu(z)Y_{\nu+1}(z) = 2/\pi z$$

we deduce after some computation

$$C_e(a, z) = \omega 2^\nu \Gamma(1 + \nu) a \\ + z^\nu C_\nu(z) \left\{ z L \left[ D_1(\nu); \frac{a^2 z^2}{4}, \frac{\xi z^2}{4} \right] + \frac{a^3 z^4}{24} Q \left[ H_1(\nu); \frac{a^2 z^2}{4}, \frac{\xi z^2}{4} \right] - 2\xi a \nu \right\} \\ + \eta z^{1+\nu} C_{\nu-1}(z) \left\{ \frac{z}{1+2\nu} L \left[ D_2(\nu); \frac{a^2 z^2}{4}, \frac{\xi z^2}{4} \right] + \frac{a^3 z^4}{48(2+\nu)} Q \left[ H_2(\nu); \frac{a^2 z^2}{4}, \frac{\xi z^2}{4} \right] - \xi a \right\}.$$

And in the same way we obtained Eq. (22) we find from this result that

$$C_s(a, z) = \omega 2^\nu \Gamma(1 + \nu) a \\ - az^\nu C_\nu(z) \left\{ \frac{a^2 z^4}{24} Q \left[ H_1(\nu); \frac{-a^2 z^2}{4}, \frac{\xi z^2}{4} \right] + 2\xi \nu \right\} \\ - \eta az^{1+\nu} C_{\nu-1}(z) \left\{ \frac{a^2 z^4}{48(2+\nu)} Q \left[ H_2(\nu); \frac{-a^2 z^2}{4}, \frac{\xi z^2}{4} \right] + \xi \right\}. \quad (25)$$

## INCOMPLETE WEBER INTEGRALS

The incomplete Weber integrals  $C_w(x, z)$  are defined for  $\operatorname{Re} \nu > -1$ ,  $|2\lambda - \arg x| < \pi/2$ ,  $z_\infty := \infty \exp i\lambda$  as follows:

$$J_w(x, z) := \int_0^z w(x, \nu; t) J_\nu(t) dt$$

$$I_w(x, z) := \int_0^z w(x, v; t) I_v(t) dt$$

$$Y_w(x, z) := \int_z^\infty w(x, v; t) Y_v(t) dt$$

$$K_w(x, z) := \int_z^\infty w(x, v; t) K_v(t) dt$$

where the subscript  $w$  indicates the presence of the kernel defined by

$$w(x, v; t) := (t/2x)^{v+1} \exp(x - t^2/4x).$$

It is shown in [1, p. 136] that  $C_{w_0}(x, z)$  may be given in terms of  $C_{e_0}(a, z)$ . Further,  $C_w(x, z)$  is completely determined by  $C_{w_0}(x, z)$  when the index  $v$  is an integer via a recursion relation [1, p. 123]. Hence,  $C_w(x, z)$  may be given in terms of  $C_{e_0}(a, z)$  or equivalently in terms of the Kampé de Fériet functions  $L, M$  or  $L, Q$ . The interested reader can easily produce formulas for  $C_w(x, z)$  in terms of these Kampé de Fériet functions. We remark only that the definition of  $C_{e_0}(a, z)$  given in [1, p. 135] uses  $\exp(-at)$  in place of  $\exp(at)$  which is used in Eq. (1).

## REDUCTION FORMULAS FOR L AND Q

Many special cases of Eqs. (11) to (18), (24) and (25) may be obtained in one form or another, provided we know a reduction formula for either  $L$  or  $Q$ . We summarize a few known relevant reduction formulas [4-8]:

$$L[\alpha, \beta; \gamma, \delta; z, z] = {}_1F_2[\alpha + \beta; \gamma, \delta; z]$$

$$L\left[D_2; \frac{z^2}{4}, \frac{z^2}{4}\right] = \frac{\sinh z}{z}$$

$$L\left[D_1; \frac{z^2}{4}, \frac{z^2}{4}\right] = \frac{2\mu}{1+2\mu} \frac{\sinh z}{z} + \frac{\cosh z}{1+2\mu}$$

$$Q\left[B_2(\mu, \mu); \frac{z^2}{4}, \frac{z^2}{4}\right] = \frac{1+\mu}{1+2\mu} \frac{4}{z^2} \left\{ \cosh z - \left(\frac{2}{z}\right)^\mu \Gamma(1+\mu) I_\mu(z) \right\}$$

$$Q\left[B_1(\mu, \mu); \frac{z^2}{4}, \frac{z^2}{4}\right] = \frac{2}{1+2\mu} \frac{1}{z} \left\{ 2\mu \frac{\cosh z}{z} + \sinh z - \left(\frac{2}{z}\right)^\mu \Gamma(1+\mu) I_{\mu-1}(z) \right\}.$$

Other properties and reduction formulas for  $L$  and  $Q$  are found in Refs. 4-8. In particular, reduction formulas for  $Q[A_1(\mu, \nu); x, x]$ ,  $Q[A_2(\mu, \nu); x, x]$ ,  $Q[B_1(\mu, \nu); x, x]$  and  $Q[B_2(\mu, \nu); x, x]$  are derived in Ref. 8.

## APPLICATIONS

Of interest in applications are the functions  $J_{\epsilon_0}(a, z)$ ,  $I_{\epsilon_0}(a, z)$ ,  $Y_{\epsilon_0}(a, z)$ , and  $K_{\epsilon_0}(a, z)$ .  $J_{\epsilon_0}(a, z)$  and  $Y_{\epsilon_0}(a, z)$  occur in problems in the theory of diffraction in optical apparatus [1, p. 227]. The function  $I_{\epsilon_0}(a, z)$  plays an important role in the study of oscillating wings in supersonic flow and arises in the study of resonant absorption in media with finite dimensions [1, p. 195].  $K_{\epsilon_0}(a, z)$  occurs when the statistical distribution of the maxima of a random function is applied to the amplitude of a sine wave in order to calculate the distribution of its ordinate. This latter distribution is of interest in the study of the scattered coherent reflected field from rough surfaces [9].

Incomplete Weber integrals of order zero occur for example in the theory of exchange processes, the analysis of drift carriers in semiconductors, and high temperature plasma physics. See Ref. 1 for further details and other applications.

Since the functions  $C_{\epsilon_0}(a, z)$  are of some importance, by using Eq. (14) and defining

$$L_1(x, y) := L \left[ \frac{1}{2}, 1; \frac{1}{2}, \frac{3}{2}; x, y \right] = L[D_1(0); x, y]$$

$$L_0(x, y) := L \left[ \frac{1}{2}, 1; \frac{3}{2}, \frac{3}{2}; x, y \right] = L[D_2(0); x, y]$$

$$Q_1(x, y) := Q \left[ 1, 1, 1; 1, 2, \frac{3}{2}; x, y \right] = Q[B_1(0, 0); x, y]$$

$$Q_0(x, y) := Q \left[ 1, 1, 1; 2, 2, \frac{3}{2}; x, y \right] = Q[B_2(0, 0); x, y]$$

we obtain

$$K_{\epsilon_0}(a, z) = z K_0(z) \left\{ L_1 \left[ \frac{a^2 z^2}{4}, \frac{z^2}{4} \right] + \frac{az}{2} Q_1 \left[ \frac{a^2 z^2}{4}, \frac{z^2}{4} \right] \right\}$$

$$+ z^2 K_1(z) \left\{ L_0 \left[ \frac{a^2 z^2}{4}, \frac{z^2}{4} \right] + \frac{az}{4} Q_0 \left[ \frac{a^2 z^2}{4}, \frac{z^2}{4} \right] \right\}$$

$$Y_{\epsilon_0}(a, z) = z Y_0(z) \left\{ L_1 \left[ \frac{a^2 z^2}{4}, \frac{-z^2}{4} \right] + \frac{az}{2} Q_1 \left[ \frac{a^2 z^2}{4}, \frac{-z^2}{4} \right] \right\}$$

$$+ z^2 Y_1(z) \left\{ L_0 \left[ \frac{a^2 z^2}{4}, \frac{-z^2}{4} \right] + \frac{az}{4} Q_0 \left[ \frac{a^2 z^2}{4}, \frac{-z^2}{4} \right] \right\}$$

$$\begin{aligned}
J_{e_0}(a, z) &= z J_0(z) \left\{ L_1 \left[ \frac{a^2 z^2}{4}, \frac{-z^2}{4} \right] + \frac{az}{2} Q_1 \left[ \frac{a^2 z^2}{4}, \frac{-z^2}{4} \right] \right\} \\
&\quad + z^2 J_1(z) \left\{ L_0 \left[ \frac{a^2 z^2}{4}, \frac{-z^2}{4} \right] + \frac{az}{4} Q_0 \left[ \frac{a^2 z^2}{4}, \frac{-z^2}{4} \right] \right\} \\
I_{e_0}(a, z) &= z I_0(z) \left\{ L_1 \left[ \frac{a^2 z^2}{4}, \frac{z^2}{4} \right] + \frac{az}{2} Q_1 \left[ \frac{a^2 z^2}{4}, \frac{z^2}{4} \right] \right\} \\
&\quad - z^2 I_1(z) \left\{ L_0 \left[ \frac{a^2 z^2}{4}, \frac{z^2}{4} \right] + \frac{az}{4} Q_0 \left[ \frac{a^2 z^2}{4}, \frac{z^2}{4} \right] \right\}.
\end{aligned}$$

The equations for  $H_{e_0}^{(1)}$  and  $H_{e_0}^{(2)}$  are the same as those for  $Y_{e_0}$  or  $J_{e_0}$  with  $Y$  or  $J$  replaced by  $H^{(1)}$  or  $H^{(2)}$ . Further, from Eq. (18) we have

$$\begin{aligned}
J_{e_0}(a, z) &= z e^{az} \left\{ L \left[ 1, \frac{1}{2}; \frac{3}{2}, 1; \frac{a^2 z^2}{4}, \frac{-z^2}{4} \right] - \frac{az}{2} L \left[ 1, \frac{1}{2}; \frac{3}{2}, 2; \frac{a^2 z^2}{4}, \frac{-z^2}{4} \right] \right\} \\
I_{e_0}(a, z) &= z e^{az} \left\{ L \left[ 1, \frac{1}{2}; \frac{3}{2}, 1; \frac{a^2 z^2}{4}, \frac{z^2}{4} \right] - \frac{az}{2} L \left[ 1, \frac{1}{2}; \frac{3}{2}, 2; \frac{a^2 z^2}{4}, \frac{z^2}{4} \right] \right\}.
\end{aligned}$$

Here we have used the properties of  $L$  that

$$L[\alpha, \beta; \gamma, \delta; x, y] = L[\alpha, \beta; \delta, \gamma; x, y] = L[\beta, \alpha; \gamma, \delta; y, x].$$

The latter results for  $C_{e_0}(a, z)$  should prove useful in numerical computation of these functions. We remark that other formulas for  $C_{e_0}(a, z)$  may be obtained from Eqs. (21) and (24).

## SUMMARY

Representations for incomplete Lipschitz-Hankel integrals of cylindrical functions using only the Kampé de Fériet functions in two variables  $L, M, Q$  are given. In addition, several known relevant reduction formulas for these functions are provided. Further, it is indicated that incomplete Weber integrals of integer order may be expressed in terms of incomplete Lipschitz-Hankel integrals of order zero. Hence incomplete Weber integrals of integer order may also be represented by the functions  $L, M, Q$ .

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